University of Idaho
High School Mathematics Competition 2024

Division I Solutions

Division I, Problem 1

Adam and Bob have been hired to type some old handwritten manuscripts into a computer. Adam would take 16 hours to do the whole job by himself, while Bob would take 12 hours to do the whole job by himself. Suppose Adam and Bob worked together for 6 hours, and then Bob left Adam to finish by himself. How much longer did Adam work for to finish the job? Assume that they don't distract each other or duplicate work and each types at the same rate they would if they were typing by themselves.

Solution: Since Bob takes 12 hours, he finishes half the job in 6 hours. This means Adam does half the jobs, and it takes him 8 hours to do so. Hence, Adam takes 2 more hours to finish the job.

Division I, Problem 2

Find the sum of the first 98 nonzero digits of 5/7 (expanded as a decimal).

Solution: Note that the decimal representation of 5/7 is

$$5/7 = 0.\overline{714285}$$

(this means that the group of 6 digits, 714285, keeps on repeating). You might have found this by long division, or some of you might have it memorized. Note that

(a) 16 and 2 are the quotient and remainder when dividing 98 by 6, respectively, so the first 98 digits will have this entire group of digits 16 times, with the first 2 digits left over at the end, and
(b) the 97th and 98th digits are 7 and 1, respectively.

Therefore, the sum of the first 98 nonzero digits of 5/7 is

$$16 \cdot (7 + 1 + 4 + 2 + 8 + 5) + 7 + 1 = 16 \cdot 27 + 8 = 440.$$

Division I, Problem 3

Find all pairs of positive integers x and y such that $x^2 - y^2 = 2024$.

Solution: Since $x^2 - y^2 = (x - y)(x + y) = 2024$, the integers x and y are either odd or even. This implies that $x - y$ and $x + y$ are both even.

The prime factorization of 2024 is $2^3 \cdot 11 \cdot 23$, and hence there are exactly four ways to write 2024 as a product of two even positive integers; namely,

$$2024 = 1012 \cdot 2 = 506 \cdot 4 = 92 \cdot 22 = 46 \cdot 44.$$

For each such pair (m, n), it suffices to find a pair (x, y) of integers satisfying the following system of linear equations.

$$\begin{cases} x + y = m \\ x - y = n. \end{cases} \tag{1}$$

Subtracting the second equation of (1) from the first, we get $x = (m+n)/2$. Substituting $x = (m+n)/2$ in the first equation gives rise to $y = (m-n)/2$. So, if $(m, n) = (1012, 2)$, $(506, 4)$, $(92, 22)$, and $(46, 44)$, then $(x, y) = (507, 505)$, $(255, 251)$, $(57, 35)$, and $(45, 1)$, respectively.

Division I, Problem 4

Let

$$x = \sqrt{2024 + \sqrt{2024 + \sqrt{2024 + \sqrt{\cdots}}}}.$$

Express x without any nested square roots.

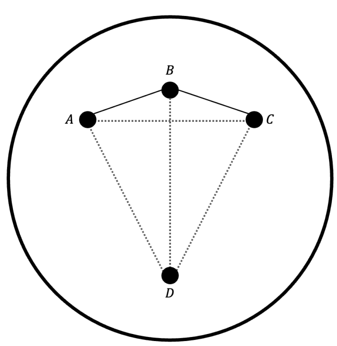
Solution: The equation can be rewritten as $x = \sqrt{2024 + x}$. Thus, x is the positive solution to the equation $x^2 = 2024 + x$ or $x^2 - x - 2024 = 0$. Hence

$$x = \frac{1 + \sqrt{1 + 4 \times 2024}}{2} = \frac{\sqrt{8097} + 1}{2}.$$

Division I, Problem 5

You wish to place four candles on a cake so that there are only 2 different distances between any 2 candles. How many different ways are there for you to do this? Make sure to justify that there are only 2 different distances for each way you discover. (Two ways are considered the same if they are similar (in the geometry sense) to each other.) You do NOT need to explain why you have found all the possibilities.

One way of placing the 4 candles is shown in the picture: segments $\overline{AB}$ and $\overline{BC}$ have one length while segments $\overline{AD}$, $\overline{BD}$, $\overline{CD}$, and $\overline{AC}$ all share a second length.



Solution:

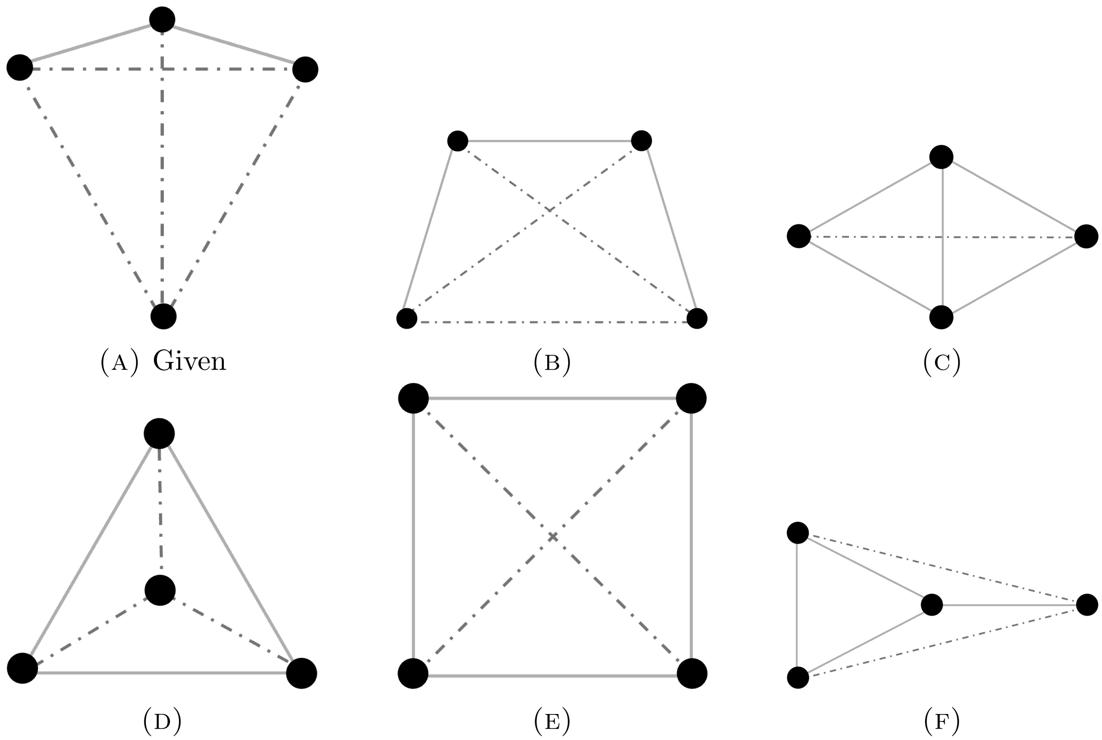


FIGURE 1. All possible configurations for the 4 points with 2 distance problem.

There are 6 possible configurations for the four candles such that there are only 2 distances between any 2 candles. All six configurations, including the example configuration are given in Figure 1.

Justification for each configuration:

For configuration 1b, a potential justification is that this is simply 4 points of a regular pentagon. Therefore, the line segments representing the vertices will all be an equivalent length and the remaining lengths will simply be the diagonals of the regular pentagon, which will all have equivalent length.

For configuration 1c, a potential justification is that this configuration is formed by joining two equilateral triangles along a common edge. Therefore, all exterior sides of the configuration will be equivalent, along with the shared base of the triangles. Then the second length is simply the connection of the two furthest points.

For configuration 1d, a potential justification is that the configuration is formed through the creation of an equilateral triangle, which by definition must have equivalent side lengths (which will be the first length). Then the fourth point is then placed at the centroid of the equilateral triangle, where the length from the centroid to each of the vertices is the same, and this then represents the second length.

For configuration 1e, a potential justification is that the configuration is formed through the creation of square. The first length is equivalent to the square side length. Then the second length is the diagonals of the square (since the diagonals have equal length).

For configuration 1f, a potential justification is that the configuration is formed through the creation of an equilateral triangle, where the first length represents the side length of the triangle. Then the fourth point is placed along the perpendicular bisector of one of sides (which we will call the base), exactly 1 triangle side length above the vertex opposite the base. The second length then are the sides of the isosceles triangle made by the fourth point and the base.

Division I, Problem 6

I have a staircase with 10 stairs. To go up the staircase, I can choose to go up one or two stairs on each step that I take. How many different ways can I climb the staircase?

Note that climbing 1+2+2+1+2+2 is different from 2+2+1+2+1+2 (and there are many other ways of arranging four 2's and two 1's that are all considered different).

Solution: For each positive integer n, we write a_n for the number of ways to climb the first n stairs of the staircase (ending on stair n). Since one can get to the n-th stair by either going to the $(n-2)$-th stair and then going up two stairs on the last step or by going to the $(n-1)$-th stair and then going up one stair on the last step, a_n is the sum of a_{n-1} and a_{n-2}:

$$a_n = a_{n-1} + a_{n-2}$$

with $a_1 = 1$ and $a_2 = 2$ (climbing $1+1$ or climbing 2). This gives the sequence

$$\begin{array}{rcccccc}
a_3 &=& a_2 + a_1 &=& 2+1 &=& 3, \\
a_4 &=& a_3 + a_2 &=& 3+2 &=& 5, \\
a_5 &=& a_4 + a_3 &=& 5+3 &=& 8, \\
a_6 &=& a_5 + a_4 &=& 8+5 &=& 13, \\
a_7 &=& a_6 + a_5 &=& 13+8 &=& 21, \\
a_8 &=& a_7 + a_6 &=& 21+13 &=& 34, \\
a_9 &=& a_8 + a_7 &=& 34+21 &=& 55, \\
a_{10} &=& a_9 + a_8 &=& 55+34 &=& 89. \\
&& & \vdots & & &
\end{array}$$

It follows that there are 89 ways to climb up the staircase with 10 stairs.

Division I, Problem 7

Consider an isosceles triangle with base length 6 and height 4. (The base is NOT one of the two equal sides.) Suppose I have a semicircle with its straight side on the base of the triangle and with its round side tangent to the two other sides of the triangle. What is the radius of this semicircle?

Solution: Call the isosceles triangle as $\triangle ABC$, where A is the top vertex. Let D be the midpoint of BC and E be the point at which AB is tangent to the semicircle. Connecting A and D, we obtain a right triangle $\triangle ABD$ with side lengths 3,4, and 5. Connecting D and E. DE is the radius of the semicircle and also the height of $\triangle ABD$ perpendicular to the base AB. Since AB has length 5 and the $\triangle ABD$ has area 6, the length of DE must be 12/5. Thus, the radius of the semicircle is 12/5.

Division I, Problem 8

Zoe plays the following one player game. On the table are 12 envelopes, numbered $1, \ldots, 12$, and the envelope numbered n has $\$n$ inside. On each turn, Zoe can take any envelope as long as that envelope has at least one other envelope labeled by one of its factors still on the table. Then all of the factors of the envelope taken are removed. Zoe then takes another turn, continuing until she can no longer take any turns (because none of the envelopes on the table have any of their factors other than themselves on the table). What is the maximum amount of money Zoe can get?

For example, Zoe can start by taking envelope 6. Once she does so, envelopes 1, 2, and 3 are also removed from the table. On the next turn, Zoe cannot take envelopes 4, 5, 7, 9, or 11 because they have no factors on the table. If Zoe takes envelope 10, then envelope 5 is also removed, and then Zoe has a choice of taking 8 or 12, either of which would end the game.

Solution: The maximum amount of money that Zoe can get is \$50. This is achieved through the selection of the following series of envelopes, as shown in Table 1.

Turn	Zoe's Chosen Envelope	Factor Envelopes Removed
1	11	1
2	10	5,2
3	9	3
4	8	4
5	12	6
Leftover	-	7

Table 1. Envelopes taken by Zoe to maximize the amount of money she gets

These turns can be somewhat reordered. For example, turn 2 and turn 3 can be switched since 10 and 9 have no common factors (since the \$1 envelope was already taken). Therefore, the solution is correct so long as the overall amount of money that Zoe has earned is \$50 and the choices are consistent.

The reason why this is the maximum amount of money is that Zoe may only select one prime number during this game, as the sole envelope factor that will be removed for a prime number is the \$1 envelope. Zoe should select the largest prime between 1 and 12, which is 11. This needs to be selected first otherwise any other envelope will remove the \$1 factor and she can no longer select this envelope. So to maximize her total amount of money, envelope 7 cannot be selected.

Therefore, there are actually 11 envelopes under consideration, and for each envelope that Zoe selects, one must go away, this means that the maximum number of turns she can take is 5. Therefore, select the 5 greatest envelopes (12, 11, 10, 9, 8) and this would yield the maximal amount of money, and Table 1 details how to select the 5 envelopes over the course of the game.